Generalised Perk–Schultz models: solutions of the Yang-Baxter equation associated with quantised orthosymplectic superalgebras

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Abstract

The Perk–Schultz model may be expressed in terms of the solution of the Yang–Baxter equation associated with the fundamental representation of the untwisted affine extension of the general linear quantum superalgebra $U_q[sl(m|n)]$, with a multiparametric co-product action as given by Reshetikhin. Here we present analogous explicit expressions for solutions of the Yang-Baxter equation associated with the fundamental representations of the twisted and untwisted affine extensions of the orthosymplectic quantum superalgebras $U_q[osp(m|n)]$. In this manner we obtain generalisations of the Perk–Schultz model.

Keywords:

1 Introduction

The Perk–Schultz model [1, 2] is well known to be exactly solvable [3]. For fixed $d > 1$, the model is defined on a square lattice where each edge can occupy one of d states. In addition to the spectral parameter the model depends on $1 + d(d-1)/2$ continuous variables, and d discrete variables which have value ± 1. One method to formulate the model and obtain the exact solution is through the R-matrix associated with the fundamental representation of the quantised untwisted affine general linear superalgebra $U_q[sl(m|n)^{(1)}]$ [4]. The exact solution follows from the fact that the R-matrix satisfies the Yang–Baxter equation. In this setting, the continuous variables are given by the deformation parameter q, as well as $d(d-1)/2$ variables associated with the Reshetikhin twist [4, 5] on the co-algebra structure. The discrete variables are associated with the $\mathbb{Z}_2$-grading of the d-dimensional vector space which affords the representation of the $U_q[sl(m|n)^{(1)}]$ superalgebra, where $m + n = d$.

Here we report the extension of this result to the case of the quantised untwisted affine superalgebra $U_q[osp(m|n)^{(1)}]$ and the twisted case $U_q[sl(m|n)^{(2)}]$ where $n = 2k$ is even in both instances. A representation theoretic approach is adopted to find R-matrices satisfying the $\mathbb{Z}_2$-graded Yang–Baxter equation (YBE)

$$R_{12}(z)R_{13}(zw)R_{23}(w) = R_{23}(w)R_{13}(zw)R_{12}(z)$$

where $R(z) \in \mathrm{End}(V(\delta_1) \otimes V(\delta_1))$ and $V(\delta_1)$ is the $(m+n)$-dimensional space for the vector representation of $U_q[osp(m|n)]$ of highest weight δ_1. The multiplication on the tensor product space is $\mathbb{Z}_2$-graded (see equation 1 in the following section). The construction of R-matrices satisfying the $\mathbb{Z}_2$-graded YBE for the general case $V(\lambda_a) \otimes V(\lambda_b)$ (where λ_a, λ_b are the highest weights of the modules) has been delineated in [6, 7]. In those works, the solutions are presented in general terms as a linear combination of elementary intertwiners, where the co-efficients are determined through tensor product graph methods. However, to have fully complete expressions it is necessary to determine also the form of the $U_q[osp(m|n)]$ invariant intertwiners which project out the submodules in the tensor

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product decomposition. Here, we will explicitly formulate R-matrices for the case $V(\delta_1)\otimes V(\delta_1)$ for $U_q[osp(m|n)]$, in both the twisted and untwisted cases by explicitly computing the elementary intertwiners. We mention that formal expressions for the solutions of the Yang–Baxter equation associated with fundamental representations of superalgebras are given in [8], which may also be used to determine explicit expressions for the R-matrices (e.g. [9]). An alternative approach is to use the Lax operator method as described in [10,11].

Once the explicit R-matrices have been obtained, we will introduce the Reshetikhin twist [5] in order to generate more general R-matrices with additional free parameters. These results can be used to obtain classes of integrable Hamiltonians describing systems of interacting fermions, with potential applications in condensed matter systems (cf. [12]).

2 The quantised orthosymplectic superalgebra $U_q[osp(m|n)]$

The quantum superalgebra $U_q[osp(m|n)]$ is a q-deformation of the classical orthosymplectic superalgebra. A brief explanation of $U_q[osp(m|n)]$ is given below, with more details to be found in [10]. Throughout we use $n=2k$ and $l=\lfloor\frac{m}{2}\rfloor$, so $m=2l$ or $m=2l+1$.

First we need to define the notation. The grading of a is denoted by $[a]$, where

$$[a]=\begin{cases}0, & a=i, \quad 1\le i\le m,\\ 1, & a=\mu, \quad 1\le\mu\le n.\end{cases}$$

We also use the symbols $\overline{a}$ and ξ_a, which are defined by:

$$\overline{a}=\begin{cases}m+1-a, & [a]=0,\\ n+1-a, & [a]=1,\end{cases} \qquad \text{and} \quad \xi_a=\begin{cases}1, & [a]=0,\\ (-1)^a, & [a]=1.\end{cases}$$

As a weight system for $U_q[osp(m|n)]$ we take the set $\{\varepsilon_i,\ 1\le i\le m\}\cup\{\delta_\mu,\ 1\le\mu\le n\}$, where $\varepsilon_{\overline{i}}=-\varepsilon_i$ and $\delta_{\overline{\mu}}=-\delta_\mu$. Conveniently, when $m=2l+1$ this implies $\varepsilon_{l+1}=-\varepsilon_{l+1}=0$. Acting on these weights, we have the invariant bilinear form defined by:

$$(\varepsilon_i,\varepsilon_j)=\delta^i_j, \quad (\delta_\mu,\delta_\nu)=-\delta^\mu_\nu, \quad (\varepsilon_i,\delta_\mu)=0, \qquad 1\le i,j\le l, \quad 1\le\mu,\nu\le k.$$

The even positive roots of $U_q[osp(m|n)]$ are composed entirely of the usual positive roots of $o(m)$ together with those of $sp(n)$, namely:

$$\begin{array}{lll}\varepsilon_i\pm\varepsilon_j, & 1\le i<j\le l, & \\ \varepsilon_i, & 1\le i\le l & \text{when } m=2l+1,\\ \delta_\mu+\delta_\nu, & 1\le\mu,\nu\le k, & \\ \delta_\mu-\delta_\nu, & 1\le\mu<\nu\le k. & \end{array}$$

The root system also contains a set of odd positive roots, which are:

$$\delta_\mu+\varepsilon_i, \qquad 1\le\mu\le k,\ 1\le i\le m.$$

Throughout this paper we choose to use the following set of simple roots:

$$\begin{aligned}\alpha_i&=\varepsilon_i-\varepsilon_{i+1}, && 1\le i<l,\\ \alpha_l&=\begin{cases}\varepsilon_l+\varepsilon_{l-1}, & m=2l,\\ \varepsilon_l, & m=2l+1,\end{cases}\\ \alpha_\mu&=\delta_\mu-\delta_{\mu+1}, && 1\le\mu<k,\\ \alpha_s&=\delta_k-\varepsilon_1.\end{aligned}$$

Note this choice is only valid for $m > 2$. Also observe that the graded half-sum of positive roots is given by:

$$\rho = \frac{1}{2}\sum_{i=1}^{l}(m-2i)\varepsilon_i + \frac{1}{2}\sum_{\mu=1}^{k}(n-m+2-2\mu)\delta_\mu.$$

In $U_q[osp(m|n)]$ the graded commutator is realised by

$$[A,B] = AB - (-1)^{[A][B]}BA$$

and tensor product multiplication is given by

$$(A\otimes B)(C\otimes D) = (-1)^{[B][C]}(AC\otimes BD). \tag{1}$$

Using these conventions, the quantum superalgebra $U_q[osp(m|n)]$ is generated by simple generators e_a, f_a, h_a subject to relations including:

$$\begin{aligned}
&[h_a, e_b] = (\alpha_a,\alpha_b)e_b, \quad [h_a, f_b] = -(\alpha_a,\alpha_b)f_b, \quad [h_a,h_b]=0,\\
&[e_a,f_b] = \delta^a_b\frac{(q^{h_a}-q^{-h_a})}{(q-q^{-1})}, \quad [e_s,e_s]=[f_s,f_s]=0.
\end{aligned}$$

We remark that $U_q[osp(m|n)]$ has the structure of a quasi-triangular Hopf superalgebra. In particular, there is a superalgebra homomorphism known as the *coproduct*, $\Delta : U_q[osp(m|n)] \to U_q[osp(m|n)]^{\otimes 2}$, which is defined on the simple generators by:

$$\begin{aligned}
\Delta(e_a) &= q^{\frac{1}{2}h_a}\otimes e_a + e_a\otimes q^{-\frac{1}{2}h_a},\\
\Delta(f_a) &= q^{\frac{1}{2}h_a}\otimes f_a + f_a\otimes q^{-\frac{1}{2}h_a},\\
\Delta(q^{\pm\frac{1}{2}h_a}) &= q^{\pm\frac{1}{2}h_a}\otimes q^{\pm\frac{1}{2}h_a}.
\end{aligned}$$

Also, $U_q[osp(m|n)]$ contains a *universal R-matrix* which satisfies, among other properties, the *Yang–Baxter equation*:

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}.$$

Here $\mathcal{R}_{ab}$ represents a copy of $\mathcal{R}$ acting on the a and b components respectively of $U\otimes U\otimes U$, where each U is a copy of the quantum superalgebra $U_q[osp(m|n)]$.

Now let End V be the space of endomorphisms of V, an $(m+n)$-dimensional vector space. Then the irreducible *vector representation* $\pi : U_q[osp(m|n)] \to \text{End } V$ acts on the $U_q[osp(m|n)]$ generators as given in Table 1, where E^a_b is the elementary matrix with a 1 in the (a,b) position and zeroes elsewhere.

Table 1: The action of the vector representation π on the simple generators of $U_q[osp(m|n)]$

α_a	$\pi(e_a)$	$\pi(f_a)$	$\pi(h_a)$
$\alpha_i, 1\le i<l$	$E^i_{i+1} - E^{\overline{i+1}}_{\overline{i}}$	$E^{i+1}_i - E^{\overline{i}}_{\overline{i+1}}$	$E^i_i - E^{\overline{i}}_{\overline{i}} - E^{i+1}_{i+1} + E^{\overline{i+1}}_{\overline{i+1}}$
$\alpha_l,\, m=2l$	$E^{l-1}_{\overline{l}} - E^{l}_{\overline{l-1}}$	$E^{\overline{l}}_{l-1} - E^{\overline{l-1}}_{l}$	$E^{l-1}_{l-1} + E^l_l - E^{\overline{l-1}}_{\overline{l-1}} - E^{\overline{l}}_{\overline{l}}$
$\alpha_l,\, m=2l+1$	$E^l_{l+1} - E^{\overline{l+1}}_{\overline{l}}$	$E^{l+1}_l - E^{\overline{l}}_{\overline{l+1}}$	$E^l_l - E^{\overline{l}}_{\overline{l}}$
$\alpha_\mu, 1\le\mu<k$	$E^\mu_{\mu+1} + E^{\overline{\mu+1}}_{\overline{\mu}}$	$E^{\mu+1}_\mu + E^{\overline{\mu}}_{\overline{\mu+1}}$	$E^{\mu+1}_{\mu+1} - E^{\overline{\mu+1}}_{\overline{\mu+1}} - E^\mu_\mu + E^{\overline{\mu}}_{\overline{\mu}}$
α_s	$E^{\mu=k}_{i=1} + (-1)^k E^{\overline{i=1}}_{\overline{\mu=k}}$	$-E^{i=1}_{\mu=k} + (-1)^k E^{\overline{\mu=k}}_{\overline{i=1}}$	$-E^{i=1}_{i=1} + E^{\overline{i}=\overline{1}}_{\overline{i}=\overline{1}} - E^{\mu=k}_{\mu=k} + E^{\overline{\mu}=\overline{k}}_{\overline{\mu}=\overline{k}}$

The solutions to the Yang–Baxter equation in a given representation of $U_q[osp(m|n)]$ can sometimes be extended to solutions of the spectral parameter dependent Yang–Baxter equation

$$R_{12}(z)R_{13}(zw)R_{23}(w) = R_{23}(w)R_{13}(zw)R_{12}(z)$$

in the affine extensions $U_q[osp(m|n)^{(1)}]$ and $U_q[gl(m|n)^{(2)}]$. In the following sections we construct such solutions for the case of the vector representation.

3 Determination of the R-matrices

The tensor product of the vector module with itself decomposes into $U_q[osp(m|n)]$ modules according to

$$V(\delta_1) \otimes V(\delta_1) = V(2\delta_1) \oplus V(\delta_1 + \delta_2) \oplus V(\dot{0})$$

except in the case $m = n$, in which case the last two irreducible modules combine to form an indecomposable V. Let

$$\mathbb{P}_V = \begin{cases} V(\delta_1+\delta_2) \oplus V(\dot{0}) & \text{for } m \neq n \\ V \text{ indecomposable} & \text{for } m = n. \end{cases}$$

Then we have a resolution of the identity as follows:

$$I = \mathbb{P}_{2\delta_1} + \mathbb{P}_V.$$

Define $\check{R}(z) = PR(z)$ where $P = \sum_{a,b}(-1)^{[b]} e^a_b \otimes e^b_a$ is the graded permutation operator. Then the Yang-Baxter equation may be rewritten as

$$\check{R}_{12}(z)\check{R}_{23}(zw)\check{R}_{12}(w) = \check{R}_{23}(w)\check{R}_{12}(zw)\check{R}_{23}(z).$$

From [6, 7] it is known that

$$\check{R} = \sum_a \rho_a(z)\mathbb{P}_a \tag{2}$$

where $\mathbb{P}_a$ denotes the $U_q[osp(m|n)]$ invariant projection operator onto the submodule $V(a)$. The coefficients $\rho_a(z)$ are determined using

$$\rho_a(z) = \left\langle \frac{C(a') - C(a)}{2} \right\rangle_{\epsilon_a \epsilon_{a'}} \rho_{a'}(z) \tag{3}$$

where

$$\langle x \rangle_\pm = \frac{1 \pm zq^x}{z \pm q^x}$$

provided the weights a, a' label adjacent vertices in the *tensor product graph* [6,7]. Here $C(a)$ denotes the eigenvalue of the second order Casimir invariant on $V(a)$ and ϵ_a the parity of the vertex associated with a. For $U_q[osp(m|n)^{(1)}]$, the tensor product graph is depicted in Figure 1 while the tensor product graph for $U_q[gl(m|n)^{(2)}]$ is given in Figure 2.

Let ψ denote the (unnormalised) basis vector for the identity module $V(\dot{0})$. Explicitly

$$\psi = \psi_0 + \psi_1$$

where

$$\psi_0 = \sum_{i=1}^{m} q^{-(\rho,\varepsilon_i)} w_i \otimes w_{\bar{i}}$$

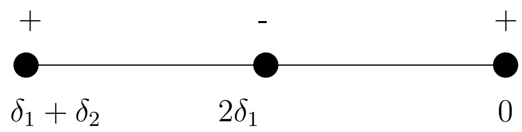

Figure 1: The untwisted tensor product graph

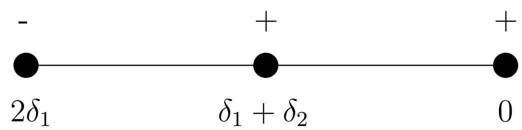

Figure 2: The twisted tensor product graph

and

$$\psi_1 = \sum_{\mu=1}^{n} -(1)^{\mu} q^{-(\rho,\delta_\mu)} w_\mu \otimes w_{\overline{\mu}}.$$

¿From equations (2) and (3), we find that for $U_q[osp(m|n)^{(1)}]$ the required R-matrix is

$$\check{R}(z) = \mathbb{P}_{2\delta_1} + \frac{1 - zq^{-2}}{z - q^{-2}} \mathbb{P}_{\delta_1+\delta_2} + \left(\frac{1 - zq^{m-n-2}}{z - q^{m-n-2}} \right) \mathbb{P}_0 \qquad (4)$$

where

$$\mathbb{P}_0 = \frac{1}{1 - [n+1-m]_q} |\psi\rangle\langle\psi|$$

and $[k]_q = \frac{q^k - q^{-k}}{q - q^{-1}}$. For $U_q[gl(m|n)^{(2)}]$ we obtain the analogous result

$$\check{R} = \mathbb{P}_{2\delta_1} + \frac{1 - zq^{-2}}{z - q^{-2}} \mathbb{P}_{\delta_1+\delta_2} + \left(\frac{1 + zq^{m-n}}{z + q^{m-n}} \right) \left(\frac{1 - zq^{-2}}{z - q^{-2}} \right) \mathbb{P}_0. \qquad (5)$$

Note that in equations (4,5) $\mathbb{P}_0$ is not defined for $m = n$. To avoid having to make separate calculations, define

$$\begin{aligned} Q &= \frac{(q - q^{-1})q^{-1}}{(q^{m-n-2} + 1)} |\psi\rangle\langle\psi| \\ &= (1 - q^{n-m}) \mathbb{P}_o. \end{aligned}$$

Then $\check{R}(z)$ can be written (and renormalised) as

$$\begin{aligned} \check{R}(z) &= \frac{z - q^{-2}}{1 - zq^{-2}} \mathbb{P}_{2\delta_1} + \mathbb{P}_{\delta_1+\delta_2} + \left(\frac{z - q^{-2}}{1 - zq^{-2}} \right) \left(\frac{1 - zq^{m-n-2}}{z - q^{m-n-2}} \right) \mathbb{P}_0 \\ &= \frac{(1 + q^{-2})(z - 1)}{1 - zq^{-2}} \mathbb{P}_{2\delta_1} + I + \frac{(z^2 - 1)}{(zq^{-2} - 1)(zq^{n-m+2} - 1)} Q \end{aligned}$$

for $U_q[osp(m|n)^{(1)}]$ and

$$\begin{aligned} \check{R}(z) &= \frac{z - q^{-2}}{1 - zq^{-2}} \mathbb{P}_{2\delta_1} + \mathbb{P}_{\delta_1+\delta_2} + \frac{1 + zq^{m-n}}{z + q^{m-n}} \mathbb{P}_0 \\ &= \frac{(1 + q^{-2})(z - 1)}{1 - zq^{-2}} \mathbb{P}_{2\delta_1} + I + \frac{(z - 1)q^{m-n}}{z + q^{m-n}} Q \end{aligned}$$

for $U_q[gl(m|n)^{(2)}]$.

In order to obtain explicit expressions for the R-matrices, it remains to determine the operator $\mathbb{P}_{2\delta_1}$. First we find the following orthogonal basis vectors for $V(2\delta_1)$:

$$
\begin{array}{cc}
q^{-1/2}w_i \otimes w_j - q^{1/2}w_j \otimes w_i, & w_\mu \otimes w_\mu, \\
q^{-1/2}w_\mu \otimes w_\nu + q^{1/2}w_\nu \otimes w_\mu, & q^{1/2}w_i \otimes w_\mu - q^{-1/2}w_\mu \otimes w_i,
\end{array}
$$

where $1 \le \mu < \nu \ne \overline{\mu} \le n$, and $1 \le i < j \ne \overline{i} \le n$. The zero weight vectors are given by the following:

$$
\begin{array}{lll}
v_i & = w_i \otimes w_{\overline{i}} - w_{\overline{i}} \otimes w_i - q^{-1}w_{i+1} \otimes w_{\overline{i+1}} + qw_{\overline{i+1}} \otimes w_{i+1}, & 1 \le i < l \\
v_s & = q^{-1}w_1 \otimes w_{\overline{1}} - qw_{\overline{1}} \otimes w_1 + (-1)^k(q^{-1}w_k \otimes w_{\overline{k}} + qw_{\overline{k}} \otimes w_k) & \\
v_\mu & = (-1)^\mu(q^{-1}w_\mu \otimes w_{\overline{\mu}} + qw_{\overline{\mu}} \otimes w_\mu + w_{\mu+1} \otimes w_{\overline{\mu+1}} + w_{\overline{\mu+1}} \otimes w_{\mu+1}), & 1 \le \mu < k \\
v_l & = w_l \otimes w_{\overline{l}} - w_{\overline{l}} \otimes w_l + \left\{ \begin{array}{ll} 0, & m = 2l \\ (q^{1/2} - q^{-1/2})w_{l+1} \otimes w_{l+1}, & m = 2l+1 \end{array} \right. &
\end{array}
$$

These, however, are not orthogonal. Instead, we complete an orthogonal dual basis for $V(2\delta_1)$ with the following orthogonal zero-weight dual vectors:

$$
\begin{aligned}
v^i &= \tilde{v}^i + \frac{D_{l-i}[k]_q}{(q+q^{-1})D_{l-k}}\Omega, \qquad 1 \le i \le l, \\
v^\mu &= \tilde{v}^\mu + \frac{[\mu]D_l}{(q+q^{-1})D_{l-k}}\Omega, \qquad 1 \le \mu < k, \\
v^s &= \frac{[k]D_l}{(q+q^{-1})D_{l-k}}\Omega,
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{v}^i &= \frac{1}{(q+q^{-1})D_l}\left\{ [i]_q \sum_{j \ge i}^{l} D_{l-j}v_j + D_{l-i}\sum_{j<i}[j]_q v_j \right\}, \\
\tilde{v}^\mu &= \frac{-1}{(q+q^{-1})[k]_q}\left\{ [\mu]_q \sum_{\nu \ge \mu}^{k-1}[k-\nu]_q v_\nu + [k-\mu]_q \sum_{\nu<\mu}[\nu]_q v_\nu \right\}
\end{aligned}
$$

and

$$
D_x = \left\{ \begin{array}{ll} \frac{q^{x-1}+q^{x-l}}{q+q^{-1}}, & m = 2l \\ \frac{q^{x-1/2}+q^{1/2-x}}{q^{1/2}+q^{-1/2}}, & m = 2l+1 \end{array} \right\} = \frac{q^{x+\frac{m}{2}-l-1} + q^{l+1-x-\frac{m}{2}}}{q^{\frac{m}{2}-l-1} + q^{l+1-\frac{m}{2}}}.
$$

It is convenient at this point to introduce the braid generator, σ:

$$
\begin{aligned}
\sigma &= q^{-1}\check{R}(0) \\
&= (q+q^{-1})\mathbb{P}_{2\delta_1} - qI + \frac{(q-q^{-1})}{q^{m-n-2}+1}|\psi\rangle\langle\psi|.
\end{aligned}
$$

Note that $\check{R}(0)$ is the same for both $U_q[osp(m|n)]$ and $U_q[gl(m|n)^{(2)}]$. After calculating $\mathbb{P}_{2\delta_1}$ and $|\psi\rangle\langle\psi|$, we find this explicit expression for the braid generator σ:

$$
\begin{aligned}
\sigma = &- \sum_{a\neq b,\bar{b}} (-1)^{[b]} E^a_b \otimes E^b_a - \sum_a (-1)^{[a]} q^{(\varepsilon_a,\varepsilon_a)} E^a_a \otimes E^a_a \\
&+ (q-q^{-1}) \left\{ \sum_{i=1}^{l} \left[\sum_{i\leq j\leq \bar{i}} q^{-(\rho,\varepsilon_i+\varepsilon_j)} E^i_j \otimes E^{\bar{i}}_{\bar{j}} + \sum_{i<j<\bar{i}} q^{-(\rho,\varepsilon_i+\varepsilon_j)} E^j_i \otimes E^{\bar{j}}_{\bar{i}} \right] \right. \\
&- \sum_{\mu\leq\nu\leq\bar{\mu}} (-1)^{\mu+\nu} q^{-(\rho,\delta_\mu+\delta_\nu)} E^\mu_\nu \otimes E^{\bar{\mu}}_{\bar{\nu}} - \sum_{\mu<\nu<\bar{\mu}} (-1)^{\mu+\nu} q^{-(\rho,\delta_\mu+\delta_\nu)} E^\nu_\mu \otimes E^{\bar{\nu}}_{\bar{\mu}} \\
&\left. + \sum_{\mu=1}^{k}\sum_{i=1}^{m} (-1)^\mu q^{-(\rho,\varepsilon_i+\delta_\mu)} (E^i_\mu \otimes E^{\bar{i}}_{\bar{\mu}} + E^\mu_i \otimes E^{\bar{\mu}}_{\bar{i}}) \right. \\
&- (q-q^{-1}) \left\{ \sum_{i<j}^{m} E^i_i \otimes E^j_j + \sum_{\mu<\nu}^{n} E^\mu_\mu \otimes E^\nu_\nu + \sum_{i=1}^{m}\sum_{\mu=1}^{k} (E^i_i \otimes E^{\bar{\mu}}_{\bar{\mu}} + E^\mu_\mu \otimes E^i_i) \right\} \\
&- \sum_{i=1}^{l} (q E^i_{\bar{i}} \otimes E^{\bar{i}}_i + q^{-1} E^{\bar{i}}_i \otimes E^i_{\bar{i}}) + \sum_{\mu=1}^{k} (q^{-1} E^\mu_{\bar{\mu}} \otimes E^{\bar{\mu}}_\mu + q E^{\bar{\mu}}_\mu \otimes E^\mu_{\bar{\mu}})
\end{aligned}
$$

Recall the relation $R(z) = P\check{R}(z)$. If we substitute in the previous equation and simplify, we obtain an expression for the R-matrices in the zero spectral parameter limit which we will denote by R':

$$
\begin{aligned}
q^{-1}R' = &- \sum_{a\neq b,\bar{b}} E^b_b \otimes E^a_a - \sum_a q^{(\varepsilon_a,\varepsilon_a)} E^a_a \otimes E^a_a \\
&- q^{-1} \sum_{i=1}^{l} (E^i_i \otimes E^{\bar{i}}_{\bar{i}} + E^{\bar{i}}_{\bar{i}} \otimes E^i_i) - q \sum_{\mu=1}^{k} (E^\mu_\mu \otimes E^{\bar{\mu}}_{\bar{\mu}} + E^{\bar{\mu}}_{\bar{\mu}} \otimes E^\mu_\mu) \\
&- (q-q^{-1}) \left\{ \sum_{i>j}^{m} E^i_j \otimes \hat{\sigma}^j_i - \sum_{\mu>\nu}^{n} E^\mu_\nu \otimes \hat{\sigma}^\nu_\mu + \sum_{i=1}^{m}\sum_{\mu=1}^{k} (E^{\bar{\mu}}_i \otimes \hat{\sigma}^i_\mu - E^i_\mu \otimes \hat{\sigma}^\mu_i) \right\}
\end{aligned}
$$

where

$$
\hat{\sigma}^a_b = E^a_b - (-1)^{[a]([a]+[b])} \xi_a \xi_b q^{(\rho,\varepsilon_b-\varepsilon_a)} E^{\bar{b}}_{\bar{a}}
$$

and

$$
\hat{\sigma}^a_a = q^{1/2(\varepsilon_a,\varepsilon_a)} E^a_a - q^{-1/2(\varepsilon_a,\varepsilon_a)} E^{\bar{a}}_{\bar{a}}.
$$

This equation simplifies further to give

$$
\begin{aligned}
q^{-1}R' = &-I - (q^{1/2}-q^{-1/2}) \sum_a (-1)^{[a]} E^a_a \otimes \hat{\sigma}^a_a \\
&- (q-q^{-1}) \left\{ \sum_{i>j}^{m} E^i_j \otimes \hat{\sigma}^j_i - \sum_{\mu>\nu}^{n} E^\mu_\nu \otimes \hat{\sigma}^\nu_\mu + \sum_{\mu=1}^{k}\sum_{i=1}^{m} (E^{\bar{\mu}}_i \otimes \hat{\sigma}^i_{\bar{\mu}} - E^i_\mu \otimes \hat{\sigma}^\mu_i) \right\}.
\end{aligned}
$$

We now rewrite $\check{R}(z)$ for $U_q[osp(m|n)^{(1)}]$ in terms of the braid generator σ.

$$\check{R}(z) = \frac{1}{(q - q^{-1}z)} \left\{ (z-1)\sigma + (q-q^{-1})zI - \frac{(q-q^{-1})z(z-1)}{(z-q^{m-n-2})} |\psi\rangle\langle\psi| \right\}.$$

Using equation (3), we can determine the normalized R-matrices as follows

$$R(z) = \frac{1}{(q - q^{-1}z)} \left\{ (z-1)q^{-1}R' + (q-q^{-1})zP - \frac{(q-q^{-1})z(z-1)}{(z-q^{m-n-2})} P|\psi\rangle\langle\psi| \right\}.$$

Explicit calculation gives the following expansion for $R(z)$ in the untwisted case:

$$\begin{aligned} R(z) = & \frac{(q-q^{-1})zP}{(q-q^{-1}z)} - \frac{(q-q^{-1})z(z-1)}{(q-q^{-1}z)(z-q^{m-n-2})} \sum_{a,b} (-1)^{[a][b]} \xi_a \xi_b q^{(\rho, \varepsilon_a - \varepsilon_b)} E^a_b \otimes E^{\bar{a}}_{\bar{b}} \\ & - \frac{(z-1)}{(q-q^{-1}z)} \left\{ I + (q^{1/2} - q^{-1/2}) \sum_a (-1)^{[a]} E^a_a \otimes \hat{\sigma}^a_a + (q-q^{-1}) \sum_{\varepsilon_a < \varepsilon_b} (-1)^{[b]} E^a_b \otimes \hat{\sigma}^b_a \right\}. \end{aligned}$$

Similarly, for $U_q[gl(m|n)^{(2)}]$ we obtain

$$\begin{aligned} R(z) = & \frac{(q-q^{-1})zP}{(q-q^{-1}z)} - \frac{(q-q^{-1})z(z-1)}{(q-q^{-1}z)(z+q^{m-n})} \sum_{a,b} (-1)^{[a][b]} \xi_a \xi_b q^{(\rho, \varepsilon_a - \varepsilon_b)} E^a_b \otimes E^{\bar{a}}_{\bar{b}} \\ & - \frac{(z-1)}{(q-q^{-1}z)} \left\{ I + (q^{1/2} - q^{-1/2}) \sum_a (-1)^{[a]} E^a_a \otimes \hat{\sigma}^a_a + (q-q^{-1}) \sum_{\varepsilon_a < \varepsilon_b} (-1)^{[b]} E^a_b \otimes \hat{\sigma}^b_a \right\}. \end{aligned}$$

We comment that although the above derivation only holds for $m > 2$, the final result holds for all m (see [10, 11]).

4 The Reshetikhin Twist

Let (A, Δ, R) denote a quasi-triangular Hopf superalgebra where Δ and R denote the coproduct and R-matrix respectively. Consider an element $F \in A \otimes A$ satisfying the properties

$$\begin{aligned} (\Delta \otimes I)F &= F_{13}F_{23}, \\ (I \otimes \Delta)F &= F_{13}F_{12}, \\ F_{12}F_{13}F_{23} &= F_{23}F_{13}F_{12}. \end{aligned}$$

Then (A, Δ^F, R^F) is also a quasi-triangular Hopf superalgebra with coproduct and R-matrix given by

$$\Delta^F = F_{12} \Delta F_{12}^{-1}, \qquad R^F = F_{21} R F_{21}^{-1}.$$

We refer to F as a *twist element*. In particular, for the case of a quantised superalgebra $U_q[g]$ Reshetikhin [5] gave the example where F is given by

$$F = \exp\Big[\sum_{b<c} (h_b \otimes h_c - h_c \otimes h_b)\phi_{bc}\Big]$$

with $\{h_b\}$ the generators of the Cartan subalgebra of $U_q[g]$ and the $\phi_{bc},\ b < c$, arbitrary complex parameters.

Applying this twist to $\check{R}(z)$, it is found that both $U_q[osp(m|n)^{(1)}]$ and $U_q[gl(m|n)^{(2)}]$ are quasi-triangular Hopf superalgebras with coproduct Δ^F as above and R-matrix in the fundamental representation given by

$$R^F(z)=\frac{(q-q^{-1})zP}{(q-q^{-1}z)}-\frac{(q-q^{-1})z(z-1)}{(q-q^{-1}z)(z-q^{m-n-2})}\sum_{a,b}(-1)^{[a][b]}\xi_a\xi_b q^{(\rho,\varepsilon_a-\varepsilon_b)}E^a_b\otimes E^{\bar{a}}_{\bar{b}}$$
$$-\frac{(z-1)}{(q-q^{-1}z)}\left\{\left(I+(q^{1/2}-q^{-1/2})\sum_a(-1)^{[a]}E^a_a\otimes\hat{\sigma}^a_a\right)\exp\Big[\sum_{b<c}2(\pi(h_c)\otimes\pi(h_b)-\pi(h_b)\otimes\pi(h_c))\phi_{bc}\Big]\right.$$
$$\left.+(q-q^{-1})\sum_{\varepsilon_a<\varepsilon_b}(-1)^{[b]}E^a_b\otimes\hat{\sigma}^b_a\right\}$$

for $U_q[osp(m|n)^{(1)}]$ and

$$R^F(z)=\frac{(q-q^{-1})zP}{(q-q^{-1}z)}-\frac{(q-q^{-1})z(z-1)}{(q-q^{-1}z)(z+q^{m-n})}\sum_{a,b}(-1)^{[a][b]}\xi_a\xi_b q^{(\rho,\varepsilon_a-\varepsilon_b)}E^a_b\otimes E^{\bar{a}}_{\bar{b}}$$
$$-\frac{(z-1)}{(q-q^{-1}z)}\left\{\left(I+(q^{1/2}-q^{-1/2})\sum_a(-1)^{[a]}E^a_a\otimes\hat{\sigma}^a_a\right)\exp\Big[\sum_{b<c}2(\pi(h_c)\otimes\pi(h_b)-\pi(h_b)\otimes\pi(h_c))\phi_{bc}\Big]\right.$$
$$\left.+(q-q^{-1})\sum_{\varepsilon_a<\varepsilon_b}(-1)^{[b]}E^a_b\otimes\hat{\sigma}^b_a\right\}$$

for $U_q[gl(m|n)^{(2)}]$. In the above formulae the representations $\pi(h_b)$ are given by Table 1. For both cases we have obtained models with $(l+k)(l+k-1)/2$ continuous variables (the ϕ_{ab}) and $m+n$ discrete variables (the grading terms $(-1)^{[a]}$: note that there must be an even number of indices a for which $[a]=1$, and also that the $\hat{\sigma}^a_b$ explicitly depend on them). These variables are in addition to the spectral parameter z. Both models may be considered as generalisations of the Perk–Schultz model.

Independently, similar results have been reported in [13].

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